1) Mettaton's x-coordinate can be calculated as

$$-1+\int_1^4 (3t^2-4)\ dt = -1+[t^3-4t]_1^4 = 50$$

2) $V=s^3$. Here, $s=3^{4/3}$. Deriving, $\frac{dV}{dt}=3s^2\frac{ds}{dt}$. Substituting, $2\cdot 3=3^{11/3}\frac{ds}{dt}$ and $\frac{ds}{dt}=2\cdot 3^{-8/3}$. $d=s\sqrt{3}$, $\frac{dd}{dt}=\sqrt{3}\frac{ds}{dt}$ and $\frac{dd}{dt}=2\cdot 3^{-13/6}$. Thus, $\{A,B,C\}=\{2,13,6\}$ and $A+B+C=21$.

3) Using the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_x^{x^2}(t^2-1)\ dt = 2x(x^4-1)-(x^2-1)$. Evaluating at $x=2$, this equals 57.

4) $f'(x)=3x^2-12x+11$. This has positive discriminant, so the function has two turning points. The function can therefore be rotated clockwise about its inflection point to where the minimum slope of the function approaches $+\infty$. The minimum slope of $f(x)$ occurs at its inflection point, which is where $6x-12=0$, or $x=2$. This slope is $f'(2)=-1$. A line with slope θ makes an angle $\operatorname{arccot}\theta$ with the y-axis, so $\alpha=45$.

5) Consider a cross-section of the cone lying on the x-y plane. Set the apex of the cone is at $(0,6)$, let the origin be the center of the base, and let the point $(8,0)$ be on the cone. Then the cone is tangent to the lines $y=6-\frac{3}{4}x$ and $y=6+\frac{3}{4}x$.

   The pyramid cross-section will be an inverted isosceles triangle with base parallel to the x-axis and apex at the origin. Let the corner of the base of the pyramid intersect the cone on the lines $y=6-\frac{3}{4}x$ and $y=6+\frac{3}{4}x$. Let the distance from the y-axis to the line $y=6-\frac{3}{4}x$ be s. This is half of a diagonal of the base, so the area of the base is $2s^2$.

   The volume of the pyramid is $V=\frac{1}{3}(2s^2)\left(6-\frac{3}{4}s\right)=4s^2-\frac{1}{2}s^3$. Deriving, $V'=8s-\frac{3}{2}s^2$. Obviously, s can't equal 0, so $s=\frac{16}{3}$ is a critical point (and a relative minimum by the Second Derivative Test). This corresponds to a volume of $\frac{1024}{27}=2^{10}/3^3$. Thus, $\{A,B\}=\{10,3\}$ and $A+B=13$.

6) The probability that Radleigh gets every question wrong is $\left(1-\frac{1}{20}\right)^{100}$. Approximating $e^{ab}=\left(1+\frac{a}{n}\right)^{bn}$ for large n, set $n=20$ and say $\left(1-\frac{1}{20}\right)^{100}\approx e^{-5}$.

7) According to Newton's Law of Cooling, the sequence formed by taking the difference between the temperature of a warmer object and the ambient temperature at constant time intervals is geometric. From the numbers given, it is seen that the temperature difference is $120\left(\frac{1}{2}\right)^{n/10}$.

   When the metal is 25 degrees, the difference is 5. The solution to $120\left(\frac{1}{2}\right)^{n/10}=5$ is between 40 and 50, corresponding to the 40-50 minute time interval.

8) Let the outside edges of the hallways lie on the positive coordinate axes and the outside corner lie on the origin. Let the length of the ladder be l. The ladder intersects the axes at $(a,0)$ and $(0,b)=(0,\sqrt{l^2-a^2})$. Thus, the ladder is the line $\frac{1}{a}x+\frac{1}{\sqrt{l^2-a^2}}y-1=0$ in the first quadrant.

In order for the ladder to have maximal length, it must pass through the inside corner, (8,8). Thus, $\frac{8}{a}+\frac{8}{\sqrt{l^2-a^2}}-1=0$ and the function $D(a)=\frac{8}{a}+\frac{8}{\sqrt{l^2-a^2}}-1$ for $a\in(0,l)$ can represent the amount of free space the ladder has. Note that $D(a)\geq 0$ for $a\in(0,l)$.

The local minimum of $D(a)$ is at $a=\frac{l}{\sqrt{2}}$. Plugging into $\frac{8}{a}+\frac{8}{\sqrt{l^2-a^2}}-1=0$, $l=\sqrt{512}$. The sum of the digits of 512 is 8.

9) The integral evaluates to $\cos k$. There are 2018 solutions to $\cos k=\frac{1}{2}$ in $[0,2018\pi]$.

10) The slope of the tangent line is $18x^2-6x+4$ at $x=2$, or 64. The x-intercept of a line with slope 64 passing through (2,32) is $\frac{3}{2}$.

11) I is false, with the following function as a counterexample.

$$f(x)=\begin{cases}|x|+1, & x\neq 0\\ 2, & x=0\end{cases}$$

II is true, by definition.

III is true, because $x=0$ fails the Vertical Line Test.

12) Let x be the number of 5-cent price reductions. The amount of money Bradley earns is $M=(1600+100x)(100-5x)=500(16+x)(20-x)=500(-x^2+4x+320)$. Deriving, $M'=500(-2x+4)$. $x=2$ is the local maximum. This corresponds to a price of \$0.90.

13) Suppose Eridan uses x gallons of pink paint. Then he uses $20-x$ gallons of purple paint. He buys $P(x)=\$\left((2x^2-x+1)+(27(20-x)+4)\right)=\$(2x^2-28x+545)$ worth of paint. $P'(x)=\$(4x-28)$, so $P(x)$ attains a local minimum at $x=7$. This corresponds to buying 13 gallons of purple paint.

14) Multiply the integrand by $\frac{\sqrt{1-x}}{\sqrt{1-x}}$ to obtain

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{1-x}{\sqrt{1-x^2}}\arcsin x\ dx=\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1-x^2}}\arcsin x\ dx-\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\frac{x}{\sqrt{1-x^2}}\arcsin x\ dx.$$

Because the first integral is odd, it can be removed. Then take $u=\arcsin x$ and $du=\frac{dx}{\sqrt{1-x^2}}$ and the integral becomes

$$-\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}u\sin u\ du=-2\int_{0}^{\frac{\pi}{3}}u\sin u\ du.$$

Integrating by parts taking $a = u$ and $db = \sin u \ du$,

$$-2\int_0^{\frac{\pi}{3}} u\sin u \ du = 2[u\cos u]_0^{\frac{\pi}{3}} + 2\int_0^{\frac{\pi}{3}} \cos u \ du.$$

Evaluating, this is equal to $\frac{\pi}{3} - \sqrt{3}$. $\{M, N\} = \{3,3\}$, so $M + N = 6$.

15) The limaçon intersects the polar origin at $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The area of the inner loop is

$$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2 + 4\cos\theta)^2 \ d\theta = 2\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta = 2\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta$$

This evaluates to $4\pi - 6\sqrt{3}$. $\{A, B, C\} = \{4,6,3\}$, so $A + B + C = 13$.

16) It is given that $\frac{V}{dV/dt} = k$. The bucket has volume 128π and $\frac{dV}{dt}$ when the bucket is full is 16, so the constant of proportionality is 8π. $\frac{V}{dV/dt} = 8\pi$. is a separable differential equation. To solve it, write $\frac{dV}{V} = \frac{1}{8\pi}dt$. Integrating, $\ln V = \frac{t}{8\pi} + C$. Since $V(0) = 8\pi$, $C = \ln 8\pi$ and the amount of water in the bucket can be represented by $\ln V = \frac{t}{8\pi} + \ln 8\pi$. Substituting in $V = 128\pi$ yields that the bucket is full at $t = 32\pi \ln 2$.

17) It is given that $v(0) = 0$ and $y(0) = 45$. $a = -10$, $v = -10t$, and $y = -5t^2 + 45$. This has a root at $t = 3$, so the ball takes 3 seconds to hit the ground.

18) $a = -10$, $v = -10t - k$, and $y = -5t^2 - kt + 45$. $y(t)$ has a root at $t = 2$, so $0 = 25 - 2k$, so $k = \frac{25}{2}$. Sameer initially throws the ball down with a speed of $\frac{25}{2}$.

19) Translate this to a Riemann sum and then into an integral:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln(i+n) - \ln n}{i+n} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{\ln\left(\frac{i}{n}+1\right)}{\frac{i}{n}+1}.$$

$$\int_0^1 \frac{\ln(x+1)}{x+1}\ dx = \int_1^2 \frac{\ln u}{u}\ du = \int_0^{\ln 2} w \ dw$$

This is equal to $\frac{\ln^2 2}{2}$.

20) The integrand is a recognizable derivative obtained through the Quotient Rule.

$$\sum_{n=1}^{\infty}\int_0^{\frac{1}{e^n}}\frac{1 - \ln x}{\ln^2 x}\ dx = -\sum_{n=1}^{\infty}\left[\frac{x}{\ln x}\right]_0^{\frac{1}{e^n}} = \sum_{n=1}^{\infty}\frac{1}{ne^n}$$

Note that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, so $\int \sum_{n=0}^{\infty} x^n\, dx = \sum_{n=1}^{\infty}\frac{x^n}{n} = \int \frac{1}{1-x}\ dx = C - \ln(1-x)$. $C = 0$ can be found by taking $x = 0$. Plugging in $x = \frac{1}{e}$ yields that $\sum_{n=1}^{\infty}\frac{1}{ne^n} = -\ln\left(1 - \frac{1}{e}\right) = 1 - \ln(e-1)$.

21) If $r = 2e^{\theta}$, then $\frac{dr}{d\theta} = 2e^{\theta}$. $\int_0^{2\pi} \sqrt{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)}\, d\theta = \int_0^{2\pi} \sqrt{8e^{2\theta}}\, d\theta = 2\sqrt{2}\int_0^{2\pi} e^{\theta}\, d\theta$. This evaluates to $2\sqrt{2}(e^{2\pi} - 1)$.

22) Let the two points on the common tangent line to the curve be $(a, a^4 - 2a^3 - 3a^2 + 3a + 7)$ and $(b, b^4 - 2b^3 - 3b^2 + 3b + 7)$. The tangent lines through these points are, respectively, $y = (4a^3 - 6a^2 - 6a + 3)x + (-3a^4 + 4a^3 + 3a^2 + 7)$ and $y = (4b^3 - 6b^2 - 6b + 3)x + (-3b^4 + 4b^3 + 3b^2 + 7)$. Because these are both representations of the exact same line, $-3a^4 + 4a^3 + 3a^2 = -3b^4 + 4b^3 + 3b^2$ and $4a^3 - 6a^2 - 6a = 4b^3 - 6b^2 - 6b$. Factoring, $-3(a^4 - b^4) + 4(a^3 - b^3) + 3(a^2 - b^2)$ and $2(a^3 - b^3) - 3(a^2 - b^2) - 3(a - b)$. Adding and dividing by 3, $-(a^4 - b^4) + 2(a^3 - b^3) - (a - b) = 0$. Since $a \neq b$, $-(a^3 + a^2b + ab^2 + b^3) + 2(a^2 + ab + b^2) - 1 = 0$. This factors to $-(a + b - 1)(a^2 - a + b^2 - b - 1) = 0$. Investigating the first term yields that $a = -1$ and $b = 2$ is a solution to the original set of equations, and the line is $y = 3 - x$. The corresponding ordinates are 4 and 1, which have a sum of 5.

23) The ellipse is $\frac{(x-3)^2}{9} + \frac{(y+4)^2}{25} = 1$, which has an area of 15π. The maximum volume of rotation would occur when $(-5,2)$ is the closest point on the axis of rotation to the centroid of the ellipse, $(3, -4)$. The distance between these two points is 10. By Theorem of Pappus, the maximum volume is $300\pi^2$.

24) $\lim\limits_{x\to 0} \frac{\sqrt{1+\sin^2(x^2)} - \cos^3(x^2)}{x^3 \tan x} = \lim\limits_{x\to 0}\left(\frac{x}{\tan x}\left(\frac{\sqrt{1+\sin^2(x^2)}-1}{x^4} + \frac{1-\cos^3(x^2)}{x^4}\right)\right) = \lim\limits_{x\to 0}\frac{\sqrt{1+\sin^2(x^2)}-1}{x^4} + \lim\limits_{x\to 0}\frac{1-\cos^3(x^2)}{x^4} = \lim\limits_{y\to 0}\frac{\sqrt{1+\sin^2 y}-1}{y^2} + \lim\limits_{y\to 0}\frac{1-\cos^3 y}{y^2}$. For the first limit, $\lim\limits_{y\to 0}\frac{\sqrt{1+\sin^2 y}-1}{y^2} = \lim\limits_{y\to 0}\frac{\sin^2 y}{y^2\left(\sqrt{1+\sin^2 y}+1\right)} = \lim\limits_{y\to 0}\frac{\sin^2 y}{y^2}\cdot\lim\limits_{y\to 0}\frac{1}{\sqrt{1+\sin^2 y}+1} = \frac{1}{2}$. For the second limit, $\lim\limits_{y\to 0}\frac{1-\cos^3 y}{y^2} = \lim\limits_{y\to 0}\frac{1-\cos y}{y^2}\cdot\lim\limits_{y\to 0}(1 + \cos y + \cos^2 y) = 3\lim\limits_{y\to 0}\frac{1-\cos y}{y^2} = 3\lim\limits_{y\to 0}\frac{\sin^2 y}{y^2}\cdot\lim\limits_{y\to 0}\frac{1}{1+\cos y} = \frac{3}{2}$. These two limits sum to 2.

25) Using the Fundamental Theorem of Calculus and deriving with respect to x, $2xf(x^2) = 2x\sin(\pi x) + \pi x^2\cos(\pi x)$. Evaluating at $x = 2$, $4f(4) = 4\pi$, so $f(4) = \pi$.

26) Let the ground level be at the height of Aaron's eyes. The screen is therefore 24 feet tall and 8 feet off the "ground." θ is defined as the viewing angle, and let α be defined as the angle between the ray extending from Aaron's eyes to the top of the screen and the ray extending from Aaron's eyes to the floor under the screen. Thus, $\tan(\theta + \alpha) = \frac{32}{x}$ and $\tan\alpha = \frac{8}{x}$.

Using the tangent addition formula, $\tan\theta = \frac{24x}{x^2+256}$. θ is maximized when $\tan\theta$ is maximized, which occurs at $x = 16$.

27) Saaketh wins if ${}^{S_1}/_{S_2}$ is in the range $\left(\frac{1}{2}, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \frac{7}{2}\right) \cup \left(\frac{9}{2}, \frac{11}{2}\right) \cup \ldots$, which can be represented as areas in the unit square in the first quadrant. The probability is the sum of the areas between $y = \frac{x}{2}$ and $y = \frac{3x}{2}$, $y = \frac{5x}{2}$ and $y = \frac{7x}{2}$, $y = \frac{9x}{2}$ and $y = \frac{11x}{2}$, etc. under the line $y = 1$, with the exception of the first region to the right of $x = 1$. Ignoring the constraint of the square, the sum of the regions under $y = 1$ is $\frac{1}{2}\cdot 1\cdot\left(\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \cdots\right) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \arctan 1 = \frac{\pi}{4}$. The area to the right of $x = 1$ is $\frac{1}{4}$. Thus, the probability Saaketh wins is $\frac{\pi-1}{4}$.

28) $A = \frac{1}{2}bh$. It is given that $b = 2$, $h = 1$, $\frac{db}{dt} = -2$, and $\frac{dh}{dt} = 1 - \sqrt{2}$. Deriving the area formula, $A' = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$. Plugging in the known values, $A' = -\sqrt{2}$. The area is decreasing at a rate of $\sqrt{2}$ per second.

29) Begin with a $u = \frac{1}{x}$ to obtain $\int_0^1 \left(\frac{\arcsin u}{u^2} - \frac{1}{u}\right) du$. Splitting the integral and then integrating by parts taking $a = \arcsin u$ and $db = \frac{du}{u^2}$ yields $\left[-\frac{\arcsin u}{u}\right]_0^1 - \int_0^1 \frac{du}{u} + \int_0^1 \frac{du}{u\sqrt{1-u^2}}$. Simplifying, this equals to $1 - \frac{\pi}{2} + \lim_{n\to 0^+} \ln n + \int_0^1 \frac{du}{u\sqrt{1-u^2}}$. Applying $u = \sin w$ produces $1 - \frac{\pi}{2} + \lim_{n\to 0^+} \ln n + \int_0^{\frac{\pi}{2}} \csc w \ dw = 1 - \frac{\pi}{2} + \lim_{n\to 0^+} \ln n + \left[\ln\left(\tan\frac{w}{2}\right)\right]_0^{\frac{\pi}{2}} = 1 - \frac{\pi}{2} + \lim_{n\to 0^+} \ln n - \lim_{n\to 0^+} \ln\left(\tan\frac{n}{2}\right) = 1 - \frac{\pi}{2} - \lim_{n\to 0^+} \frac{\tan\frac{n}{2}}{n} = 1 - \frac{\pi}{2} + \ln 2$. Thus, $\{A, B, C\} = \{1,2,2\}$, and $A + B + C = 5$.

30) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$.

1. D
2. C
3. E
4. B
5. C
6. B
7. D
8. B
9. B
10. D
11. D
12. C
13. D
14. A
15. D
16. A
17. C
18. B
19. A
20. A
21. D
22. C

23. D
24. B
25. C
26. C
27. C
28. A
29. B
30. B